SEMI-DISCRETE ANALYSIS OF ELECTROMAGNETIC FIELDS FOR DIFFERRENT SHAPES OF EXCITATION

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ABSTRACT

The paper deals with numerical aspects of the FE-analysis in time domain by means of semi-discrete method. The method provides analytical solution in time, so the time-stepping may be omitted. The method seems to be numerically very effective, however, the produced matrices are dense. The well known time-stepping method requires a new matrix decomposition, while adapting the time-step. On the contrary, the proposed method suites very well for the cases of non-uniform time steps, particularly while using adjoint models [1].

The solutions utilising semi-discrete method were compared with ordinary time-stepping. The exemplary models are taken from non-destructing testing apparatus utilising eddy-currents. The currents exciting the NDT-probe take often the form of single sinusoidal, or rectangular impulse. For the aim of modelling of NDT-probes the semi-discrete solution for different shapes of excitation is shown.

Index Terms – electromagnetic fields, time-domain analysis.

1. INTRODUCTION

The proposed semi-discrete method allows us to obtain time-domain solution without time-stepping. For space discretization we use usual finite elements of first order. The semi-discrete method delivers analytical and continuous solution for any given time of analysis, which takes a form of exponential functions.

2. FE-TS ANALYSIS

Electromagnetic field diffusion into conducting region may be for two-dimensional models described utilizing vector magnetic potential $\boldsymbol{A}$:

$$\nabla^2 \boldsymbol{A} - \mu\gamma \frac{\partial \boldsymbol{A}}{\partial t} = -\mu_0 \cdot \boldsymbol{i}(t) = -\boldsymbol{f}(t). \qquad (1)$$

Approximating (1) with finite elements and applying time-dependent elements we bring the solution to commonly used **F**inite **E**lement – **T**ime **S**tepping method, leading to the following system of linear equations:

$$\left(\Theta\cdot\boldsymbol{K} + \frac{\boldsymbol{M}}{\Delta t}\right)\cdot\boldsymbol{A}(t_{i+1}) = \\ = \Theta\cdot\boldsymbol{f}(t_{i+1}) + (1-\Theta)\cdot\boldsymbol{f}(t_i) + \left(\frac{\boldsymbol{M}}{\Delta t} - (1-\Theta)\cdot\boldsymbol{K}\right)\cdot\boldsymbol{A}(t_i), \qquad (2)$$

with: $\boldsymbol{K}$ – stiffness matrix, $\boldsymbol{M}$ – mass matrix, Δt – time step and $0{,}5 \le \Theta \le 1$ defines the differential scheme of time stepping method. The effective solution of (2) is as long possible, as the time step Δt remains constant. For example, when applying this method to adjoint model to calculate sensitivities of electromagnetic field [2], [3], the time moments, at which both, original and adjoint model are analyzed, should coincide as shown in Fig.1:

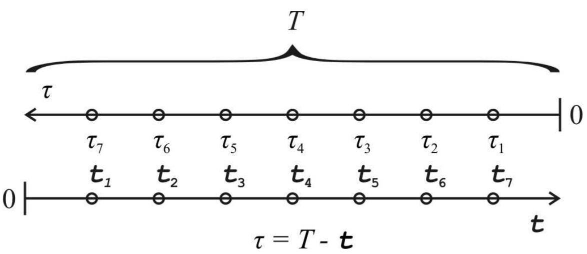

Figure 1. Time-stepping for original and adjoint model.

This requirement causes the large number of iterative steps and very long computational time. This is the reason, why the semi-discrete method has been developed.

3. SEMI-DISCRETE FINITE ELEMENT ANALYSIS

The inhomogeneous diffusion equation we are solving has the form of

$$[\boldsymbol{K}]\{\boldsymbol{A}(t)\} + [\boldsymbol{M}]\frac{\partial \boldsymbol{A}(t)}{\partial t} = \{\boldsymbol{f}(t)\}. \qquad (3)$$

The transient component of $\boldsymbol{A} = \boldsymbol{A}_s + \boldsymbol{A}_u$ results from homogeneous equation:

$$[\boldsymbol{K}]\{\boldsymbol{A}_s(t)\} + [\boldsymbol{M}]\frac{\partial \boldsymbol{A}_s(t)}{\partial t} = 0\,, \qquad (4)$$

which solution is

$$\{A_s(t)\}=\{C\}\cdot\exp\left(-t\cdot[M]^{-1}[K]\right),$$
$$\text{with: } \{C\}-\text{constant of integration.} \tag{5}$$

The constant $\boldsymbol{C}$ depends on initial value $\boldsymbol{A}(0)$, which is zero in our case, and on the excitation shape. The solutions for different excitations $\boldsymbol{f}$ are shown below.

4. UNIT-STEP EXCITATION

However the current owning the shape of unit-step $\mathbf{1}(t)$ can not squeeze into the coil, it is handy approximation of the real state.

The steady-state response of the magnetic vector potential $\boldsymbol{A}_u$ in this case has the form:

$$\{A_u(t)\}=[K]^{-1}\{f(t)\}=[K]^{-1}\cdot\mathbf{1}(t)\,. \tag{6}$$

The constant of integration $\boldsymbol{C}$ can be evaluated from the initial condition

$$\{A_s(0)\}+\{A_u(0)\}=\{A(0)\}=\mathbf{0} \tag{7}$$

and the semi-discrete solution for vector magnetic potential takes the form:

$$\{A(t)\}=\left([\mathbf{1}]-\exp\left(-t\,[M]^{-1}[K]\right)\right)\cdot[K]^{-1}\{\mathbf{1}(t)\}\,. \tag{8}$$

When the inversion of mass matrix is necessary, we can not apply (8) for the whole region. It has to be subdivided into a conductive part "1" and a non-conductive part "2":

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{Bmatrix} A_1(t) \\ A_2(t) \end{Bmatrix}+\begin{bmatrix} M_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{Bmatrix} \dfrac{\partial}{\partial t}A_1(t) \\ \mathbf{0} \end{Bmatrix}=\begin{Bmatrix} \mathbf{0} \\ f_2 \end{Bmatrix} \tag{9}$$

We assume that excitation currents are located only in a non-conducting part of the region. Then we obtain the following matrix equation for the conducting region:

$$\left([K_{11}]-[K_{12}][K_{22}]^{-1}[K_{21}]\right)\{A_1(t)\}+[M_{11}]\{\partial A_1(t)/\partial t\}=$$
$$=-[K_{12}][K_{22}]^{-1}\{f_2(t)\},\ \text{or:} \tag{10}$$
$$[K_c]\{A_1(t)\}+[M_{11}]\{\partial A_1(t)/\partial t\}=\{f_c(t)\}.$$

The similarity of (10) to (3) allows exploitation of solution (8). Fig. 2 shows the comparison of magnetic vector potential calculated by FE-TS Eqn.(2) using Galerkin-scheme (Θ=2/3) versus semi-discrete solution. The both solutions were obtained using simple two-dimensional model described below.

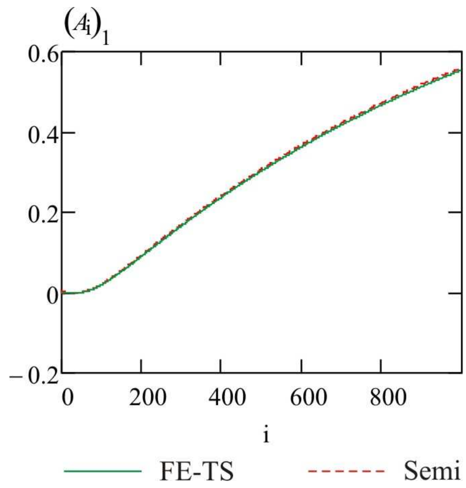

Figure 2. Comparison of $\boldsymbol{A}$ *in node "1" for* t_i.

5. DESCRIPTION OF TEST MODEL

For the purpose of testing a very simple model, shown in Fig. 3, was chosen. The model exhibits Cartesian symmetry. It consists of the conducting region with material parameters $1/6\cdot\gamma\cdot\mu_0=1$ and $\mu_r=1$. Excitation is produced by a line current of 1 A/m density. A 2D-model is driven with 0.5 A currents directed to nodes "21" and "22". The element matrices for a 2D–case are [4]:

$$K_e=\frac{1}{4\Delta^e\mu_r^e}\begin{bmatrix} b_i^2+c_i^2 & b_ib_j+c_ic_j & b_ib_k+c_ic_k \\ b_ib_j+c_ic_j & b_j^2+c_j^2 & b_jb_k+c_jc_k \\ b_ib_k+c_ic_k & b_jb_k+c_jc_k & b_k^2+c_k^2 \end{bmatrix},$$
$$M_e=\frac{\mu_0^e\gamma^e\Delta^e}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},\ f_e=\frac{\Delta^e}{3}\mu_0^e\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \tag{11}$$

with: $b_i=y_j-y_k$, $c_i=x_k-x_j$,

and Δ^e – area of the element e.

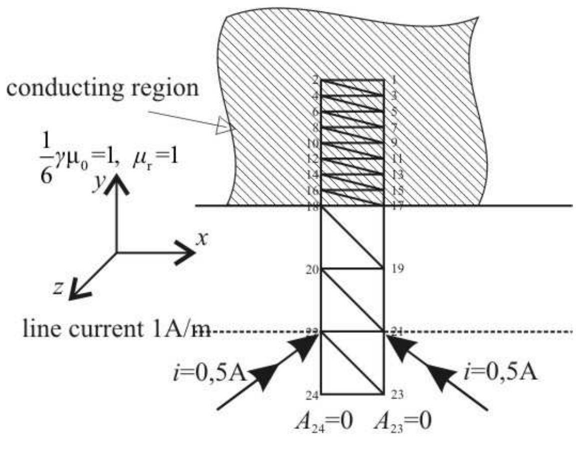

Figure 3. 2D test model.

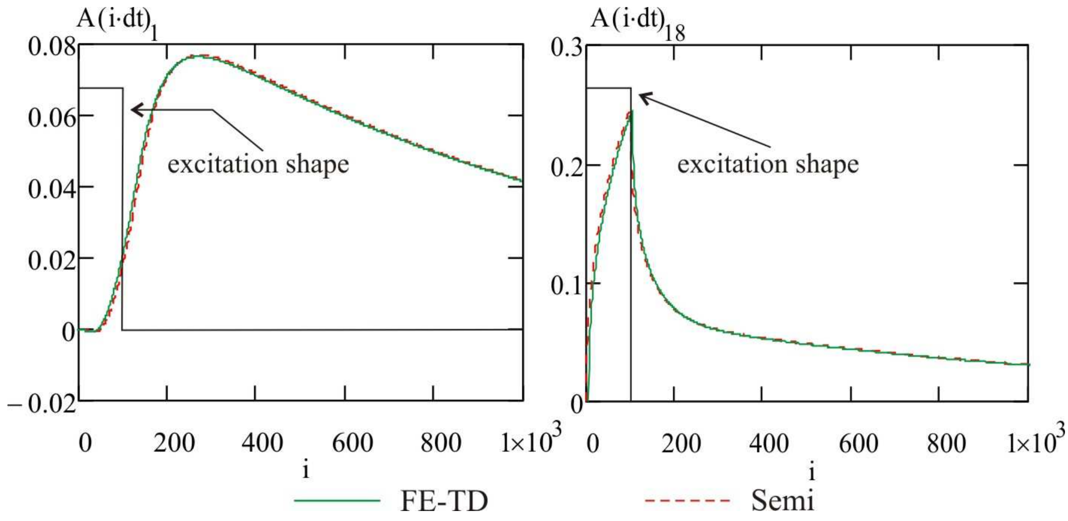

Figure 5. The solutions for rectangular excitation current shape.

6. HARMONIC EXCITATION

For harmonic excitation we have the following inhomogeneous differential equation:

$$[\boldsymbol{K}]\{\boldsymbol{A}(t)\}+[\boldsymbol{M}]\frac{\partial \boldsymbol{A}(t)}{\partial t}=\{\boldsymbol{I}_{\mathrm{m}}\sin(\omega t)\}\ . \qquad (12)$$

The steady-state response $\boldsymbol{A}_{\mathrm{u}}$ may be evaluated in frequency-domain with standard FEM:

$$[\boldsymbol{K}]\{\underline{\boldsymbol{A}}_{\mathrm{u}}\}+\mathrm{j}\omega[\boldsymbol{M}]\{\underline{\boldsymbol{A}}_{\mathrm{u}}\}=\{\underline{\boldsymbol{I}}\}\,, \qquad (13)$$

and then, we can retrieve the time function of steady state response

$$\begin{aligned}&\{\underline{\boldsymbol{A}}_{\mathrm{u}}\}=([\boldsymbol{K}]+\mathrm{j}\omega[\boldsymbol{M}])^{-1}\{\underline{\boldsymbol{I}}\},\\&\{\boldsymbol{A}_{\mathrm{u}}(t)\}=\mathrm{imag}\left(([\boldsymbol{K}]+\mathrm{j}\omega[\boldsymbol{M}])^{-1}\cdot\{\underline{\boldsymbol{I}}\}\cdot\exp(\mathrm{j}\omega t)\right).\end{aligned} \qquad (14)$$

Assuming zero initial condition we evaluate now the constant of integration $\boldsymbol{C}$

$$\begin{aligned}&\{\boldsymbol{C}\}+\mathrm{imag}\left(([\boldsymbol{K}]+\mathrm{j}\omega[\boldsymbol{M}])^{-1}\cdot\{\underline{\boldsymbol{I}}\}\right)=\boldsymbol{0},\\&\{\boldsymbol{C}\}=-\mathrm{imag}\left(([\boldsymbol{K}]+\mathrm{j}\omega[\boldsymbol{M}])^{-1}\cdot\{\underline{\boldsymbol{I}}\}\right).\end{aligned} \qquad (15)$$

So, the semi-discrete solution for transient magnetic vector potential with harmonic excitation takes the form

$$\begin{aligned}&\{\boldsymbol{A}(t)\}=\mathrm{imag}\left(([\boldsymbol{K}]+\mathrm{j}\omega[\boldsymbol{M}])^{-1}\cdot\{\underline{\boldsymbol{I}}\}\cdot\exp(\mathrm{j}\omega t)\right)-\\&-\mathrm{imag}\left(([\boldsymbol{K}]+\mathrm{j}\omega[\boldsymbol{M}])^{-1}\cdot\{\underline{\boldsymbol{I}}\}\right)\cdot\exp\left(-t\cdot[\boldsymbol{M}]^{-1}[\boldsymbol{K}]\right)=\\&=\mathrm{imag}(\underline{\boldsymbol{A}}_{\mathrm{u}}\cdot\exp(\mathrm{j}\omega t))-\mathrm{imag}(\underline{\boldsymbol{A}}_{\mathrm{u}})\cdot\exp\left(-t\cdot[\boldsymbol{M}]^{-1}[\boldsymbol{K}]\right).\end{aligned} \qquad (16)$$

The comparison of semi-discrete solution with FE-TS method is shown in Fig.4. It concerns the node number (1), accordingly to Fig.3.

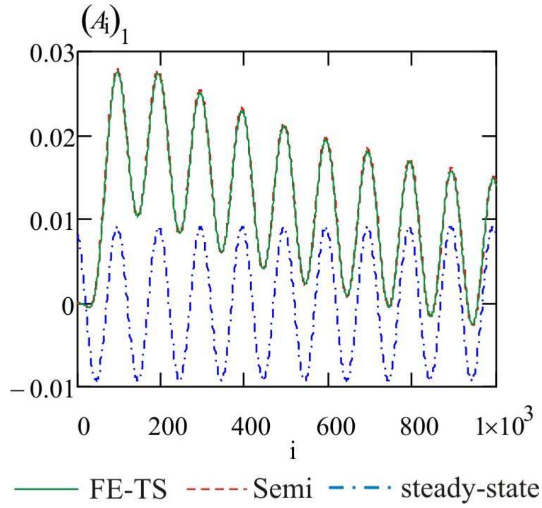

Figure 4. Comparison of magnetic vector potential for harmonic excitation.

7. EXCITATION WITH THE RECTANGULAR IMPULSE

The solution for rectangular impulse was achieved as superposition of solutions for two unit-step impulses:

$$\{\boldsymbol{A}(t)\}=\begin{cases}([\mathbf{1}]-\mathrm{e}^{-t[\boldsymbol{M}_{11}]^{-1}[\boldsymbol{K}_{\mathrm{c}}]})\cdot[\boldsymbol{K}_{\mathrm{c}}]^{-1}\{\mathbf{1}(t)\}\ \text{if}\quad t\le T,\\(\mathrm{e}^{-(t-T)[\boldsymbol{M}_{11}]^{-1}[\boldsymbol{K}_{\mathrm{c}}]}-\mathrm{e}^{-t[\boldsymbol{M}_{11}]^{-1}[\boldsymbol{K}_{\mathrm{c}}]})\cdot[\boldsymbol{K}_{\mathrm{c}}]^{-1}\{\mathbf{1}(t)\}\\\text{otherwise.}\end{cases} \qquad (17)$$

In the Fig. 5 the solutions obtained for the node inside conducting material (node 1) and for the one on the border to the air (node 18), are shown. The shape of excitation current was also placed there.

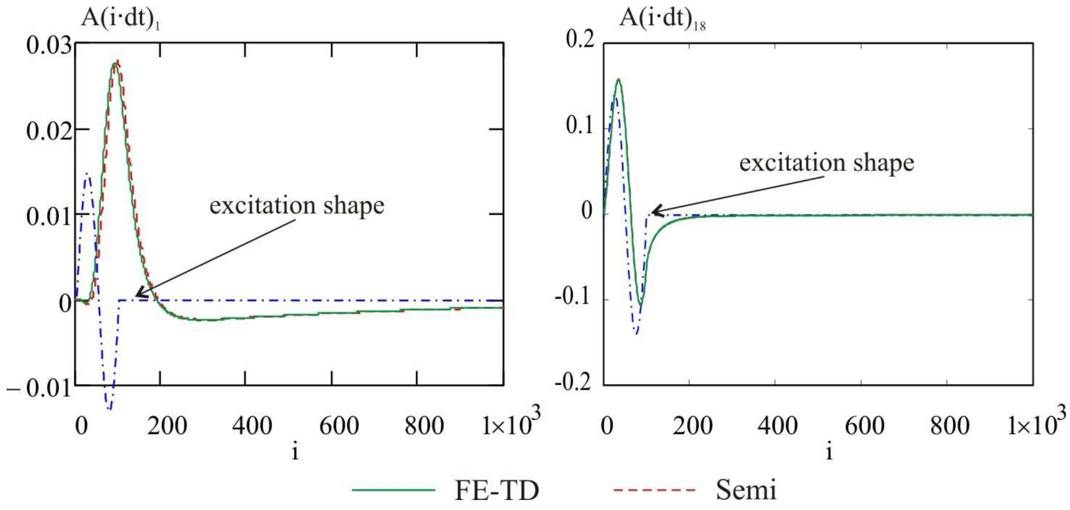

Figure 6. Comparison of magnetic vector potential for single sinusoidal current impulse.

8. SINGLE SINUSOIDAL PULSE

The solution for single, sinusoidal current pulse was derived in the same manner, using superposition of two solutions given by Eqn.(16):

$$\{A(t)\}=\begin{cases}\text{imag}\left(\underline{\boldsymbol{A}}_u\cdot e^{j\omega t}-(\underline{\boldsymbol{A}}_u)\cdot e^{-t\cdot[\boldsymbol{M}]^{-1}[\boldsymbol{K}]}\right) \text{ if } t\le T,\\ \text{imag}\left((\underline{\boldsymbol{A}}_u)\cdot e^{-(t-T)\cdot[\boldsymbol{M}]^{-1}[\boldsymbol{K}]}-(\underline{\boldsymbol{A}}_u)\cdot e^{-t\cdot[\boldsymbol{M}]^{-1}[\boldsymbol{K}]}\right)\\ \text{otherwise.}\end{cases} \qquad (18)$$

The comparison shown in Fig. 6 reveals good agreement of solutions obtained using semi-discrete method versus this from classical FE-time-stepping. For the node number (1) we can observe the delay of the response.

9. CONCLUSIONS

The proposed method allows the semi-discrete evaluation of electromagnetic fields, without time-stepping. When using the standard FE-TS method for the original and adjoint model (in the backward time), a practically constant time step needs to be applied. However, if the algorithm is applied to solution of inverse problems by means of the gradient method [1], useful information is delivered not only by first time steps but also by advanced time points. Next, we have to meet a compromise between the size of time step and the number of steps [2]. The aforementioned problem vanishes when using the semi-discrete time-domain sensitivity analysis. A drawback of this method is matrices which are losing their symmetry and are no more banded. All calculations in this work were carried out with fully assigned matrices.

Comparison of the efficiency of the semi-discrete method with classical FE-TS [3] shows, that despite of high demand for memory, the described method may compete in relation to finite elements with the time stepping.

10. REFERENCES

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